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Classical exchange algebra in Liouville theory on a Riemann surface

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Abstract. Following Zograf and Takhtajan, we use Schottky uniformization of $g>1$ Riemann surfaces to obtain the Liouville equation and its general solution on the covering space of a Riemann surface. Formal apparatus of inverse scattering theory is employed to compute the exchange algebra of the chiral solutions to a second-order linear differential equation. The results depend on $3g$-3 complex parameters.

1. Introduction

Ever since Polyakov's seminal work [1] on 2D quantum gravity, the study of both classical and quantum Liouville theory [2] has attracted much attention. The interests are really twofold. On the one hand, even though the matrix models [3] proved very successful, several aspects, including the problematic strong coupling region, of the continuous model still seem mysterious. On the other hand, Liouville theory, as the simplest case of Toda theories, is conformal invariant which, among other CFTs, is intimately related to quantum group [4] through its quantum exchange algebra, as shown by Gervais in [5].

In the previous studies, either in conformal field theory or in quantum gravity, most of the work considered only topologically trivial 2D manifolds (there are of course several exceptions, see [6]). It is evident that Liouville theory on general Riemann surfaces is of crucial importance. In this respect, Zograf and Takhtajan [7] have proved a quite intriguing result concerning a relation between the accessory parameters in the uniformatization theory and the action of the Liouville equation on an arbitrary Riemann surface of genus $g>1$. The Liouville action proposed by Zograf and Takhtajan has the property that it coincides with the usual one when restricted to the fundamental domain of the Schottky group, while the extra terms bring up dependence on the $3g$-3 Schottky space parameters. Note that the Liouville equation on the covering space of a Riemann surface is obtained as the Euler-Lagrange equation of the proposed action. In addition, according to an old corollary of the uniformization theorem, every hyperbolic metric of constant negative curvature on a Riemann surface, as an induced Poincaré metric, satisfies the Liouville equation on the covering space. The Liouville field determined by this metric obeys a delicate transformation law under the Schottky group.

In this paper starting from the prescribed Liouville theory on the covering space of a Riemann surface, we carry out a study of the classical inverse problem for the Liouville equation on an arbitrary Riemann surface, paying attention to the similarity between the general solution of the Liouville equation on a Riemann surface and that on a plane. In the next section, we briefly recall some necessary mathematical facts concerning the Schottky group and uniformization. The way the Liouville equation and its general solution on a Riemann surface come into play is also explained. In section 3, we introduce the change of variables, diagonalize the monodromy matrices by a new set of dynamical variables and compute the basic Poisson brackets. As a by-product, we obtain a standard free oscillator expression for Liouville fields. In section 4, we bring together our results of calculating exchange algebra between the classical analogues of chiral vertex operators. Finally, in section 5, we summarize this paper with several comments.

2. Mathematical preliminaries

A marked Riemann surface of genus $g>1$, is a Riemann surface M with $x_0 \in M$ taken as fixed, together with a choice of a set of generators α_i, β_i, $i=1,\dots,g$ of the funadmental group $\pi_1(M, x_0)$. Given this, a dissection of M can be performed by cutting off M along $2g$ homology cycles, starting from the point x_0. The result is a planar polygon in a subregion of (extended) complex plane. The reverse procedure of dissection, which identifies edges of the polygon by the prescribed generators α_i, β_i, is what the famous uniformization theorem amounts to. The mathematically rigorous way of uniformizing a surface is to take a covering space Ω, and a set of covering maps. Let G be the automorphism group of the covering transformations, then Ω/G is a Riemann surface (of genus $g>1$). In this paper we always choose as G the Schottky group Σ, which is a (discontinuous) group consisting of strictly loxodromic transformations. The corresponding covering space is the extended complex plane $\hat{C}=CP^1\cup(\infty)$, i.e. Σ maps $\hat{C}$ into $\hat{C}$, via fractional linear transformation γ:

$$\gamma(w)=\frac{aw+b}{cw+d} \qquad w\in\hat{C} \qquad ad-cb=1. \tag{2.1}$$

Every strictly loxodromic transformation has only two distinct fixed points [8], $\gamma(\xi_{1,2})=\xi_{1,2}$, $\xi_1\neq\xi_2$, so every generator L of the Schottky group can be uniquely (up to an overall SL(2, C) transformation) determined by its fixed points and a constant multiplier:

$$\frac{L(w)-\xi_1}{L(w)-\xi_2}=\lambda\frac{w-\xi_1}{w-\xi_2} \qquad 0<|\lambda|<1. \tag{2.2}$$

The condition for $|\lambda|$ is such that ξ_1 and ξ_2 are repulsive and attractive, respectively. It is easily seen that

$$\xi_{1,2}=\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c} \qquad \lambda+\frac{1}{\lambda}=(a+d)^2-2. \tag{2.3}$$

Let us choose a set of free generators L_i, $i=1,\dots,g$, of Σ (such a Σ is named the marked Schottky group). The corresponding marked Riemann surface can be obtained as follows. Σ maps $\hat{C}$ into $\hat{C}$. Exlcuding a finite set of limiting points of Σ, a subset Ω in $\hat{C}$ is called the region of discontinuity of the (properly discontinuous) group Σ, Σ acts on Ω freely, and according to the uniformization theorem, Ω/Σ is the desired Riemann surface ($g>1$). Σ maps the interior of the isometric circle $|cz+d|=1$, to the exterior of the isometric circle $|cz-a|=1$. So, the fundamental domain of Σ in $\hat{C}$ is a region $D=\bigcup_{i=1}^{g} D_i \cup D_i'$, bounded by $2g$ disjoint Jordan curves A_i, A_i', $i=1,\ldots,g$ (see figure 1). An important fact is $A_i'=-L_i(A_i)$, with action of L_i reversing the orientation of A_i.

Equation (2.3) implies that the Schottky group can be parametrized by $3g-3$ (minus three because of the conjugation by SL(2, C)) complex parameters. More than this, there is a natural isomorphism from the marked Schottky group $\{\Sigma, L_i\}$ to a subset in C^{3g-3}:

$$\{\Sigma, L_i\} \mapsto (\xi_1^{(i)}, \xi_2^{(i)}, \lambda_i) \in S_g \subset C^{3g-3} \tag{2.4}$$

here S_g is the Schottky space [9] connected to Teichmuller space in a suitable way.

In the following sections, the so-called Fuchsian equation and its independent solutions on the covering space of Riemann surfaces will be referred to. So we shall summarize a few facts about them here.

The Fuchsian equation is a second-order linear differential equation satisfied by a meromorphic differential on M of order $-\frac{1}{2}$ [10]:

$$y''+\tfrac{1}{2}\mathcal{S}[u]y=0 \tag{2.5}$$

where $\mathcal{S}[u]$ is the Schwartz differential defined as

$$\mathcal{S}[u]=\frac{u'''}{u'}-\frac{3}{2}\left(\frac{u''}{u'}\right)^2 \tag{2.6}$$

with the property that $\mathcal{S}[\gamma]=0$, for γ a fractional linear transformation.

Two linearly independent solutions y_1, y_2 to equation (2.5) exist and their ratio y_1/y_2 solves the Schwarz equation (2.6). In each analytic coordinate patch, the system of equations (2.5) and (2.6) is trivially satisfied by analytic functions. However, it becomes non-trivial globally, due essentially to the transformation law of $\mathcal{S}[u]$ under the coordinate change $z\to w$:

$$\mathcal{S}[u(w)]=\mathcal{S}[u(z)](w')^2+\mathcal{S}[w(z)]. \tag{2.7}$$

Since equation (2.5) is covariant (with $u(w)$ transforms as differential of $-\frac{1}{2}$ order) under coordinate transformation, it makes sense to speak of solutions of equation (2.5) defined globally on a Riemann surface M. These are topologically inequivalent solutions indexed by $2g$ homology bases of M. It was proved in [10] that such solutions depend on $3g-3$ complex parameters (we ignore here the dependence of the Schwartz connection on accessory parameters which, however, is needed to fix up the arbitrariness of Schwartz connections on M.) We will see below that these $2g$ topologically inequivalent

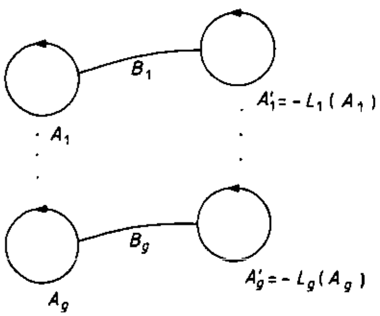

Figure 1. The fundamental domain of the Schottky group. Circles are identified by L_i to obtain a g-handled surface.

solutions of the Fuchsian equation are crucial for linearizing the boundary conditions obeyed by the Liouville field variables.

Let us return to Schottky uniformization. Remember that every Riemann surface of $g>1$ is equipped with a (hyperbolic) metric of constant negative curvature. Let it be of the form $ds^2 = h\,dw\,d\bar{w}$; its Gauss curvature is

$$K = -\frac{1}{h}\partial_w\partial_{\bar{w}} \ln h. \tag{2.8}$$

Choosing $h = e^{\varphi}$, the requirement of constant negative curvature then leads to the Liouville equation:

$$\partial_w\partial_{\bar{w}}\varphi = e^{\varphi} \tag{2.9}$$

(here we have normalized K to be -1). The advantage of Schottky uniformization is that it gives rise to a smooth solution of equation (2.9) [11], $\varphi(w)$ on Ω, such that

$$e^{\varphi(w)} = \frac{|\partial_w f(w)|^2}{(\mathrm{Im}\, f(w))^2} \qquad w \in \Omega \subset \hat{C} \tag{2.10}$$

with transformation law under the Schottky group:

$$e^{\varphi(Lw)} = e^{\varphi(w)}|L'(w)|^{-2} \qquad L \in \Sigma. \tag{2.11}$$

The f in equation (2.10) is a conformal mapping from Ω to the upper half-plane H (f is a multivalued function), and its Schwartz differential coincides with the energy-momentum tensor constructed from $\varphi(w)$:

$$\mathcal{S}[f] = \varphi_{ww} - \tfrac{1}{2}\varphi_w^2 = T_{\mathrm{Liou}}. \tag{2.12}$$

Based on the above observations, Zograf and Takhtajan proposed the following action of the Liouville system on the covering space Ω of a Riemann surface:

$$S(\varphi) = \frac{1}{2}\int_0 (\varphi_w\varphi_{\bar{w}} + e^{\varphi})\,d^2w - \frac{i}{2}\sum_{i=2}^{g}\int_{A_i}\left(\varphi\frac{\bar{L}_i''}{\bar{L}_i'}\,d\bar{w} - \varphi\frac{L_i''}{L_i'}\,dw\right)$$
$$+\frac{i}{2}\sum_{i=2}^{g}\int_{A_i}(\log|L_i'|^2)\frac{\bar{L}_i''}{\bar{L}_i'}\,d\bar{w} - 4\pi\sum_{i=2}^{g}\log|l_i|^2 \tag{2.13}$$

$$l_i = \frac{1-\lambda_i}{\sqrt{\lambda_i}\,(\xi_1^i - \xi_2^i)}.$$

It is evident that variation of $S(\varphi)$ gives rise to the Liouville equation (2.9). The reason for the last three terms in the action (2.13) is to make the action independent of the choice of fundamental domain D in Ω. Thus it is well defined and universal on Ω. Note that in this way the Liouville field $\varphi(w)$ acquires additional dependence on $3g-3$ parameters in S_g.

3. The classical inverse problem method for the Liouville equation

In the preceding section, we have seen that the Liouville equation can be consistently put on the covering space of the Riemann surface, with Liouville fields transforming definitely under the Schottky group. We also learnt that topologically inequivalent solutions to the Fuchsian equation can be parametrized by $3g-3$ complex parameters.

Now we are ready to carry out a study of the classical periodic problem for the Liouville system on a Riemann surface. One of the important concepts in the inverse problem method is the definition of equal time Poisson brackets. The 'time' in complex coordinate w can be taken as $t=\ln|w|$ by relation: $\ln w=t+\mathrm{i}x$, $\ln w=t-\mathrm{i}x$. In the (x, t) coordinate system, the Liouville equation takes the form

$$\varphi_{tt}-\varphi_{xx}-\frac{2}{\beta}\,\mathrm{e}^{\beta\varphi}=0 \tag{3.1}$$

here the coupling constant β of φ (x, t) is explicitly introduced. Being exactly the same form (except for a minus sign before the last term in the left-hand side, which is responsible for the negative curvature we are choosing) as in flat space-time, equation (3.1) admits the same zero curvature expression. We refer the reader to [12] for the notation.

From the zero curvature expression of equation (3.1), we read off the classical r-matrix in our case

$$r=\begin{pmatrix}0&0&0&0\\0&-\tilde{\alpha}&2\tilde{\alpha}&0\\0&0&-\tilde{\alpha}&0\\0&0&0&0\end{pmatrix}\qquad \tilde{\alpha}=\frac{\beta^2}{8}. \tag{3.2}$$

An important notion in the inverse scattering theory is the transition matrix $T(x, x_0)$, which is defined as

$$T(x, x_0)=P\exp\left(\int_{x_0}^{x} U(x', t)\,\mathrm{d}x'\right) \tag{3.3}$$

here P denotes path ordering, and $U(x', t)$ is a field-dependent matrix which appeared in the linearized auxiliary equation (see [12]).

By ultralocality of the T-matrix, it is easily checked that $T(x, x_0)$ satisfies the fundamental Poisson bracket:

$$\{T(x, x_0)\overset{\otimes}{,} T(x, x_0)\}=[r, T(x, x_0)\otimes T(x, x_0)] \tag{3.4a}$$

here we adhere to the convention for the tensor product

$$(A\otimes B)_{ij,mn}=A_{im}B_{jn}$$

and

$$\{A\overset{\otimes}{,} B\}_{ij,mn}=\{A_{im}, B_{jn}\}.$$

For our purposes of calculation, it is convenient to use the alternative form of equation (3.4*a*):

$$\{T(x)\overset{\otimes}{,} T(y)\}=(T(x, y)\otimes 1)[r, T(y)\otimes T(y)]\qquad x>y \tag{3.4b}$$

with a similar equation for $x<y$.

Equation (3.4*b*) is a manifestation of the following property of $T(x, y)$:

$$T(x)=T(x, y)\,T(y). \tag{3.5}$$

In terms of matrix elements of $T(x)$, we introduce a pair of functions $u(x)$, $v(x)$ which serve as substitutes for canonical pair of Liouville field variables φ, π,

$$u(x)=\frac{T_{12}(x)}{T_{11}(x)}\qquad v(x)=\frac{T_{22}(x)}{T_{21}(x)}. \tag{3.6}$$

Using equations (3.6) and (3.4*b*), we compute the Poisson bracket of u, v as follows:

$$\begin{aligned}&\{u(x), u(y)\} = \tilde{\alpha}\ \mathrm{sgn}(x-y)[u(x)-u(y)]^2 - \tilde{\alpha}[u^2(x)-u^2(y)]\\&\{v(x), v(y)\} = -\tilde{\alpha}\ \mathrm{sgn}(x-y)[v(x)-v(y)]^2 - \tilde{\alpha}[v^2(x)-v^2(y)]\\&\{u(x), v(y)\} = -2\tilde{\alpha}[u^2(x)-u(x)v(y)].\end{aligned} \tag{3.7}$$

Now we are in the position to apply the standard methods of inverse scattering to the problem at hand, i.e. calculation of the Poisson bracket relations for the canonically transformed variables (see below).

Due to the smoothness of the Liouville fields φ and π on Ω, they are periodic along each non-trivial cycle on M. From this one readily deduces the following periodicity of the transition matrix $T(x)$:

$$T(x) \mapsto T(x)\, T^{A^i\ (\text{or}\ A_i')} \tag{3.8}$$

x taken around cycles A or A, $i = 1 \ldots g$.

The matrices T^{A_i}, $T^{A_i'}$ are monodromy matrices along A_i and A_i', respectively.

Denote the matrix elements of T^{A_i}, $T^{A_i'}$ by

$$T^{A_i} = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \qquad T^{A_i'} = \begin{pmatrix} \alpha_i' & \beta_i' \\ \gamma_i' & \delta_i' \end{pmatrix} \tag{3.9}$$

it is easily checked from equations (3.8) and equation (3.6) that

$$\begin{aligned}&u(x + \text{around } A_i) = T_p^{A_i}(u(x)) \qquad v(x + \text{around } A_i) = T_p^{A_i}(v(x))\\&u(x + \text{around } A_i') = T_p^{A_i'}(u(x)) \qquad v(x + \text{around } A_i') = T_p^{A_i'}(v(x)).\end{aligned} \tag{3.10}$$

In the above expression, $T_p^{A_i}$ and $T_p^{A_i'}$ act on u, v by fractional linear transformation, and are defined by

$$T_p^{A_i} = \begin{pmatrix} \delta_i & \beta_i \\ \gamma_i & \alpha_i \end{pmatrix} \qquad T_p^{A_i'} = \begin{pmatrix} \delta_i' & \beta_i' \\ \gamma_i' & \alpha_i' \end{pmatrix}. \tag{3.11}$$

The change of variables φ, $\pi \mapsto u, v$ is not the whole story, since u, v obey complicated 'boundary conditions' (equation (3.10)). We need to linearize the system further by diagonalizing the monodromy matrices T^{A_i} (the other set of monodromy matrices $T^{A_i'}$ will be shown later to be related to T^{A_i} by simple relations.) The diagonalization of T^{A_i} can be carried out using the $2g$ fixed points z_{1i}, z_{2i}, of matrices $T_p^{A_i}$ (remember $T_p^{A_i}$ is hyperbolic, as usual):

$$T_p^{A_i}(z_{1,2i}) = z_{1,2i} \qquad i = 1 \ldots g \tag{3.12}$$

with

$$z_{1,2i} = \frac{\delta_i - \alpha_i \pm \sqrt{(\alpha_i + \delta_i)^2 - 4}}{2\gamma_i}. \tag{3.13}$$

Diagonalization of T^{A_i} is accomplished by the change of variables,

$$\begin{aligned}&u(x) \mapsto u_i(x) = \frac{u(x) - z_{1,i}}{u(x) - z_{2,i}}\\&v(x) \mapsto v(x) = \frac{v(x) - z_{1i}}{v(x) - z_{2i}}.\end{aligned} \tag{3.14}$$

Now the new set of variables obey the standard quasiperiodic boundary conditions:

$$u(x+\text{ around } A_i) = e^{-2p_i}\, u_i(x)$$
$$v(x+\text{ around } A_i) = e^{-2p_i}\, v_i(x) \tag{3.15}$$

here

$$e^{p_i} = \delta_i - z_{1i}\gamma_i \qquad e^{-p_i} = \delta_i - z_{2i}\gamma_i \tag{3.16}$$

further,

$$e^{2p_i} + e^{-2p_i} = (\delta_i + \alpha_i)^2 - 2. \tag{3.17}$$

Comparing with equation (2.3), we see a one-to-one correspondence between (z_{1i}, z_{2i}, p_i) and $(\xi_1^i, \xi_2^i, \lambda_i)$ is natural. This implies that we obtain a set of parameters isomorphic to the Schottky space mentioned in section 2.

What about the 'boundary conditions' for $u(x)$ along the A_i' cycles? To find the answer, note that $A_i' = -L_i(A_i)$. Let us use the transformation law of $\varphi(w)$ (equation (2.11)) (and a similar one for $\pi(w)$ of course) to compute $T(L_i(x))$. It follows that

$$T(L_i(x), L_i(x_0)) = P\exp\left[\int_{x_0}^{x} U(x', t)\, dx' + \int_{x_0}^{x} K_i(x', t)\, dx'\right]$$
$$K_i(x', t) = \frac{1}{2}\left(\frac{L_i^{-1''}(t+ix')}{L_i^{-1'}(t+ix')} + \frac{\bar{L}_i^{-1''}(t+ix')}{\bar{L}_i^{-1'}(t+ix')}\right)\sigma_3 \tag{3.18}$$

where the prime denotes differentiation w.r.t complex variable w, and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We need to find out the explicit relation between $T(L_i(x))$ and $T(x)$. Note the field independence of $K_i(x, t)$; we are able to use the gauge covariance of the Lax pair equation, taking account of the singularity of $K_i(x, t)$. So let

$$T(L_i(x), L_i(x_0)) = G_i(x, x_0)\tilde{T}_i(x, x_0) \tag{3.19}$$

such that $G_i(x, x_0)$ satisfies

$$G_i^{-1}(x, x_0)\partial_x G_i(x, x_0) = G_i^{-1}(x, x_0) K_i(x, t) G_i(x, x_0). \tag{3.20}$$

Applying equations (3.18)-(3.20), we have

$$\partial_x \tilde{T}_i(x, x_0) = G_i^{-1}(x, x_0) U(x, t) G_i(x, x_0). \tag{3.21}$$

In terms of the formal solution to equation (3.21), we can express $T(L_i(x), L_i(x_0))$ as follows:

$$\begin{aligned} T(L_i(x), L_i(x_0)) &= G_i(x, x_0) P\exp\left[\int_{x_0}^{x} G_i^{-1}(x', x_0) U(x', t) G_i(x', x_0)\, dx'\right] \\ &= T(x, x_0) G_i(x_0, x_0). \end{aligned} \tag{3.22}$$

Note that $G_i(x_0, x_0)$ is no longer an identity matrix since $K_i(x, t)$ is singular. Rather, it contributes to the general solution to equation (3.20):

$$\begin{aligned} G_i(x, x_0) &= P\exp\left[\int_{x_0}^{x} K_i(x', t)\, dx' + \int_{A_i'''} K_i(x', t)\, dx'\right] \\ &= P\exp\left[\int_{x_0}^{x} K_i(x', t)\, dx'\right]\begin{pmatrix} e^{-4m\pi} & 0 \\ 0 & e^{4m\pi} \end{pmatrix}. \end{aligned} \tag{3.23}$$

Here A_i^m means taking around the cycle A_i for m times, $m \in Z$ (counting different directions). In obtaining equation (3.23), we have taken account of the residue of K_i.

It follows from the above calculation that

$$T^{A'_i} = T^{A_i}\begin{pmatrix} e^{-4m\pi} & 0 \\ 0 & e^{4m\pi} \end{pmatrix}. \tag{3.24}$$

Therefore, we have also linearized the boundary condition of $u(x)$ along A'_i cycles. In the mean time we linearize the A_i cycle boundary conditions, i.e.

$$\begin{aligned} u_i(x + \text{around } A'_i) &= e^{-2p_i+8m\pi}\, u_i(x) \\ v_i(x + \text{around } A'_i) &= e^{-2p_i+8m\pi} v_i(x). \end{aligned} \tag{3.25}$$

As usual in the linearized problem of the integrable systems, the parameters, such as $z_{1,2i}$ and p_i used to diagonalize the monodromy matrices, are all dyanmical variables; they should have non-trivial Poisson brackets, so, in evaluating the Poisson brackets between linearized field variables u_i, v_i, one should take account of their contribution. Happily, as we have checked in detail, the Poisson brackets among the parameters $z_{1,2i}$ and p_i all vanish, even though their Poisson brackets with u, v definitely do not. After a tedious but direct computation, we obtain (for $i = j$):

$$\begin{aligned} \{u_i(x), u_i(y)\} &= \tilde{\alpha}\,\mathrm{sgn}(x-y)[u_i(x) - u_i(y)]^2 + \tilde{\alpha}\coth(p_i)[u_i^2(x) - u_i^2(y)] \\ \{v_i(x), v_i(y)\} &= -\tilde{\alpha}\,\mathrm{sgn}(x-y)[v_i(x) - v_i(y)]^2 - \frac{\tilde{\alpha}(v_i(x) - v_i(y))}{e^{-2p_i} - 1} \\ &\quad \times [(v_i(x) - 1)(v_i(y) - e^{-2p_i}) + (v_i(y) - 1)(v_i(x) - e^{-2p_i})] \\ \{u_i(x), v_i(y)\} &= \frac{-2\tilde{\alpha}}{1 - e^{-2p_i}}\, u_i(x)(v_i(y) - 1)(v_i(y) + e^{-2p_i}). \end{aligned} \tag{3.26}$$

By introducing

$$P_{1i}(x) = u_i''(x)/u_i'(x) \qquad P_{2i}(x) = v_i''(x)/v_i'(x) \tag{3.27}$$

one easily sees that $P_{1,2i}(x)$ satisfy the free field Poisson brackets:

$$\begin{aligned} \{P_{1i}(x), P_{1i}(y)\} &= 4\tilde{\alpha}\delta'(x-y) \qquad & \{P_{1i}(x), P_{2i}(y)\} &= 0 \\ \{P_{2i}(x), P_{2i}(y)\} &= -4\tilde{\alpha}\delta'(x-y) \qquad & \{p_i, P_{1,2i}(x)\} &= 0 \end{aligned} \tag{3.28}$$

and obey the periodic boundary conditions along cycles A_i, A'_i. We note that $P_{1,2i}$ have the following property:

$$\begin{aligned} \int_{A_i} P_{1i}(x)\,dx &= \int_{A_i} P_{2i}(x)\,dx = -2p_i \\ \int_{A'_i} P_{1i}(x)\,dx &= \int_{A'_i} P_{2i}(x)\,dx = -2p_i + 8m\pi. \end{aligned} \tag{3.29}$$

Obviously p_i serves as the zero mode in the Fourier expansions of $P_{1,2i}$. Thus we arrive at the essence of the free field formulation of the Liouville system.

4. Classical exchange algebra

In this section we shall compute the exchange algebra by using equation (3.26) for general i, j. Recall that in section 2 we explained that on a Riemann surface of genus

$g>1$, the Fuchsian equation (2.5) admits $2g$ topologically inequivalent solutions. Denote by $\psi_{1,2i}$ these $2g$ solutions which are expressible by the linearized variables $u_i(x)$ and their derivatives $u_i'(x)$, as follows:

$$\psi_{1i}(x)=[u_i'(x)]^{-1/2} \qquad \psi_{2i}(x)=u_i(x)[u_i'(x)]^{-1/2} \qquad i=1\dots g. \tag{4.1}$$

We are interested in the Poisson brackets between $u_i(x)$, $u_i(y)$ at $i\neq j$ (the case $i=j$ was treated in section 3, equation (3.26)). A tedious but direct calculation shows

$$\{u_i(x),u_j(y)\} = \tilde{\alpha}[z_{ii}^{12}z_{jj}^{12}]^{-1}\{\mathrm{sign}(x-y)[\Delta_{ij}^{(1)}(x,y)-\Delta_{ji}^{(1)}(y,x)]^2+\Delta_{ij}^{(2)}(x,y)-\Delta_{ji}^{(2)}(y,x)\} \tag{4.2}$$

here, $z_{ij}^{\mu\nu}=z_{\mu i}-z_{\nu j}$, $\mu,\nu=1,2$;

$$\begin{aligned}\Delta_{ij}^{(1)}(x,y)&=[z_{1i}-z_{2i}u_i(x)][1-u_j(y)]\\ \Delta_{ij}^{(2)}(x,y)&=\coth(p_j)\Delta_{ij}^{(1)}(x,y)[z_{ij}^{11}-z_{ij}^{21}u_i(x)+(z_{ij}^{12}-z_{ij}^{22}u_i(x))u_j(y)].\end{aligned} \tag{4.3}$$

Using equations (4.2) and (4.3), we are able to find the Poisson brackets of $\psi_{i,2i}$, i.e. the exchange algebra. Let us denote by $\Psi(x)=(\psi_{11}(x),\psi_{21}(x),\ \psi_{12}(x),\psi_{22}(x),\dots,\psi_{1g}(x),\psi_{2g}(x))$ the row vector spanned by the g pairs of $\psi_{1,2i}(x)$ defined in equation (4.1).

Then the exchange algebra has the following form;

$$\{\Psi(x)\overset{\otimes}{,}\Psi(y)\}=(\Psi(x)\otimes\Psi(y))Q \tag{4.4}$$

where Q is a $4g^2\times4g^2$ matrix with field-independent matrix elements. Written in component form:

$$\{\psi_l(x),\psi_m(y)\}=\sum_{l',m'}\psi_{l'}(x)\psi_{m'}(y)Q_{l'm',lm}. \tag{4.5}$$

The explicit expression for $Q_{l'm',lm}$ is as follows:

(i) *l, m being simultaneously odd:*

$$\begin{aligned}Q_{l'm',lm}=\tfrac{1}{2}\tilde{\alpha}(z^{12}_{(l-1)/2+1,(l-1)/2+1}z^{12}_{(m-1)/2+1,(m-1)/2+1})^{-1}(&a^{oo}_{(l-1)/2+1,(m-1)/2+1}\delta_{l',l}\delta_{m',m}\\ &+b^{oo}_{(l-1)/2+1,(m-1)/2+1}\delta_{l',l+1}\delta_{m',m}\\ &+c^{oo}_{(l-1)/2+1,(m-1)/2+1}\delta_{l',l}\delta_{m',m+1}+d^{oo}_{(l-1)/2+1,(m-1)/2+1}\delta_{l',l+1}\delta_{m',m+1})\\ a^{oo}_{i,j}&=\mathrm{sgn}(x-y)z_{ji}^{22}z_{ji}^{11}-\tfrac{1}{2}[\coth(p_j)(z_{1i}+z_{2i})z_{jj}^{12}-i\leftrightarrow j)\\ b^{oo}_{i,j}&=-2\,\mathrm{sgn}(x-y)z_{ji}^{22}z_{ji}^{12}+\coth(p_j)z_{2i}z_{jj}^{12}-\coth(p_i)(z_{1j}z_{ji}^{22}+z_{2j}z_{ji}^{12})\\ c^{oo}_{i,j}&=-2\,\mathrm{sgn}(x-y)z_{ji}^{21}z_{ji}^{22}+\coth(p_j)(z_{ij}^{12}z_{2i}+z_{1i}z_{ij}^{22})-\coth(p_i)z_{2j}z_{ii}^{12}\\ d^{oo}_{i,j}&=2[\mathrm{sgn}(x-y)(z_{ji}^{22})^2-\coth(p_j)z_{2i}z_{ij}^{22}+\coth(p_i)z_{2j}z_{ji}^{22}]\end{aligned} \tag{4.6}$$

(ii) *l odd, m even:*

$$\begin{aligned}Q_{l'm',lm}=\tfrac{1}{2}\tilde{\alpha}(z^{12}_{(l-1)/2+1,(l-1)/2+1}z^{12}_{m/2,m/2})^{-1}(&a^{oe}_{(l-1)/2+1,m/2}\delta_{l',l}\delta_{m',m-1}\\ &+b^{oe}_{(l-1)/2+1,m/2}\delta_{l',l+1}\delta_{m',m-1}\\ &+c^{oe}_{(l-1)/2+1,m/2}\delta_{l',l}\delta_{m',m}+d^{oe}_{(l-1)/2+1,m/2}\delta_{l',l+1}\delta_{m',m})\\ a^{oe}_{i,j}&=-2\,\mathrm{sgn}(x-y)z_{ji}^{21}z_{ji}^{12}+\coth(p_j)(z_{2i}z_{ij}^{11}+z_{1i}z_{ij}^{21})+\coth(p_i)z_{ij}z_{ii}^{12}\\ b^{oe}_{i,j}&=(-2)[\mathrm{sgn}(x-y)(z_{ji}^{12})+\coth(p_j)z_{2i}z_{ij}^{21}+\coth(p_i)z_{ij}z_{ji}^{12}]\\ c^{oe}_{i,j}&=-\mathrm{sgn}(x-y)(z_{ji}^{22}z_{ji}^{11}+z_{ji}^{22}z_{ji}^{21})+\tfrac{1}{2}(\coth(p_j)(z_{1i}+z_{2i})z_{jj}^{12}-i\leftrightarrow j)\\ d^{oe}_{i,j}&=2\,\mathrm{sgn}(x-y)z_{jj}^{12}z_{ji}^{22}-\coth(p_j)z_{2i}z_{jj}^{12}+\coth(p_i)(z_{2j}z_{ji}^{12}+z_{ij}z_{ji}^{22})\end{aligned} \tag{4.7}$$

(iii) l even, m odd:

$$Q_{l'm',lm} = \frac{1}{2}\tilde{\alpha}(z^{12}_{l/2,l/2} z^{12}_{(m-1)/2+1,(m-1)/2+1})^{-1}(a^{eo}_{l/2,(m-1)/2+1}\delta_{l',l-1}\delta_{m',m} + b^{eo}_{l/2,(m-1)/2+1}\delta_{l',l}\delta_{m',m} + c^{eo}_{l/2,(m-1)/2+1}\delta_{l',l-1}\delta_{m',m+1} + d^{eo}_{l/2,(m-1)/2+1}\delta_{l',l}\delta_{m',m+1})$$

$$a^{eo}_{i,j} = 2\,\mathrm{sgn}(x-y) z^{21}_{ji} z^{11}_{ji} - \coth(p_j) z_{1i} z^{12}_{jj} - \coth(p_i)(z_{2j} z^{11}_{ji} + z_{1j} z^{21}_{ji}) \tag{4.8}$$

$$b^{eo}_{i,j} = -\mathrm{sgn}(x-y)(z^{22}_{ji} z^{21}_{ji} z^{12}_{ji}) + \tfrac{1}{2}(\coth(p_j)(z_{1i} + z_{2i}) z^{12}_{jj} - i \leftrightarrow j)$$

$$c^{eo}_{i,j} = 2[-\mathrm{sgn}(x-y)(z^{21}_{ji})^2 + \coth(p_j) z_{1i} z^{12}_{ij} + \coth(p_i) z_{2j} z^{21}_{ji}]$$

$$d^{eo}_{i,j} = 2\,\mathrm{sgn}(x-y) z^{22}_{ji} z^{21}_{ji} - \coth(p_j)(z_{2i} z^{12}_{ij} + z_{1i} z^{22}_{ij}) + \coth(p_i) z_{2j} z^{12}_{ii}$$

(iv) l, m simultaneously even:

$$Q_{lm',lm} = \tfrac{1}{2}\tilde{\alpha}(z^{12}_{l/2,l/2} z^{12}_{m/2,m/2})^{-1}(a^{ee}_{l/2,m/2}\delta_{l',l-1}\delta_{m',m-1} + b^{ee}_{l/2,m/2}\delta_{l',l}\delta_{m',m-1} + c^{ee}_{1/2,m/2}\delta_{l',l-1}\delta_{m',m} + d^{ee}_{l/2,m/2}\delta_{l',l}\delta_{m',m}) \tag{4.9}$$

$$a^{ee}_{i,j} = 2[\mathrm{sgn}(x-y)(z^{11}_{ji})^2 + \coth(p_j) z_{1i} z^{11}_{ij} - \coth(p_i)(z_{ij} z^{11}_{ji}]$$

$$b^{ee}_{i,j} = -[2\,\mathrm{sgn}(x-y) z^{12}_{ji} z^{11}_{ji} + \coth(p_j)(z_{1i} z^{21}_{ij} + z_{2i} z^{11}_{ij}) + \coth(p_i)(z_{ij} z^{12}_{ii}]$$

$$c^{ee}_{i,j} = -2\,\mathrm{sgn}(x-y) z^{11}_{ji} z^{21}_{ji} + \coth(p_j) z_{1i} z^{12}_{jj} + \coth(p_i)(z_{2j} z^{11}_{ji} + z_{ij} z^{21}_{ji})$$

$$d^{ee}_{i,j} = \mathrm{sgn}(x-y) z^{12}_{ji}(z^{21}_{ji} - \tfrac{1}{2}(\coth(p_j)(z_{1i} + z_{2i}) z^{12}_{jj} - i \leftrightarrow j).$$

5. Concluding remarks

To end this paper, a few comments are in order.

As we have just seen, Liouville theory on a high-genus Riemann surface shares most of the integrability characteristics in its flat space version. As equations (3.27) and (3.28) suggest, there exists a set of g pairs of free field oscillator variables all satisfying simple PB relations. This appears in accordance with the fact that on a Riemann surface of genus g, there are g zero modes for the holomophic (1, 0) forms.

Usually, in the flat space version of exchange algebra, one considers exchange of the two chiral vertices corresponding to the two linearly independent solutions to the Fuchsian equation. As we have argued in section 2, on $g > 1$ Riemann surfaces, however, the Fuchsian equation has $2g$ topologically inequivalent solutions corresponding to $2g$ non-trivial monodromies around homoglogy cycles. In view of possible implication to representation of a braid group on a Riemann surface, it is reasonable to take these topologically inequivalent solutions as independent, thus justifying the appearance of the $4g^2 \times 4g^2$ exchange matrix in equation (4.4).

The interesting depence of our results on the $3g-3$ complex parameters in Schottky space has the following interpretation. Because Schottky space is (locally) isomorphic to Teichmuller space, which is the space of metric (complex) structures, the dependence on these 'moduli' parameters implies that different complex structures in 2D manifolds should give rise to different exchange algebra structures. This is also natural, since different complex structures are related by quasiconformal transformation. One should not expect 'moduli' independent physical results from an intrinsically conformal invariant theory.

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